

Pion light-cone wave function and pion distribution amplitude in the Nambu–Jona-Lasinio model

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We compute the pion light-cone wave function and the pion quark distribution amplitude in the Nambu–Jona-Lasinio model. We use the Pauli-Villars regularization method and as a result the distribution amplitude satisfies proper normalization and crossing properties. In the chiral limit we obtain the simple results, namely $\varphi_\pi(x) = 1$ for the pion distribution amplitude, and $\int d^2k_\perp \Psi_\pi(x, \vec{k}_\perp) k_\perp^2 = \langle \vec{k}_\perp^2 \rangle = -M \langle \bar{u}u \rangle / f_\pi^2$ for the second moment of the pion light-cone wave function, where M is the constituent quark mass and f_π is the pion decay constant. After the QCD Gegenbauer evolution of the pion distribution amplitude good end-point behavior is recovered, and a satisfactory agreement with the analysis of the experimental data from CLEO is achieved. This allows us to determine the momentum scale corresponding to our model calculation, which is close to the value $Q_0 = 313$ MeV obtained earlier from the analogous analysis of the pion parton distribution function. The value of $\langle \vec{k}_\perp^2 \rangle$ is, after the QCD evolution, around $(400 \text{ MeV})^2$. In addition, the model predicts a linear integral relation between the pion distribution amplitude and the parton distribution function of the pion, which holds at the leading-order QCD evolution.

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I. INTRODUCTION

The study of high-energy exclusive processes [1] provides a convenient tool of learning about the quark substructure of hadrons. In this limit the total amplitude factorizes into a hard contribution, computable from perturbative QCD, and a soft matrix element which requires a non-perturbative treatment. From the point of view of chiral symmetry breaking a particularly interesting process is provided by the $\gamma^* \rightarrow \gamma^* \pi^0$ transition form factor. For real photons its normalization is fixed by the anomalous breaking of chiral symmetry by the $\pi^0 \rightarrow \gamma\gamma$ decay. In addition, in the limit of large photon virtualities, factorization allows us to define the leading-twist pion distribution amplitude as a low energy matrix element whose normalization is fixed by the pion weak-decay constant, a spontaneous chiral symmetry breaking feature of the QCD vacuum. It seems obvious that such a process offers a unique opportunity not only to learn about the interplay between high and low energies, but also to understand the relation between the spontaneous and the anomalous chiral symmetry breaking. Radiative logarithmic corrections to the pion distribution amplitude (PDA) can be easily implemented through the QCD evolution equations [2, 3], which yield for $Q^2 \rightarrow \infty$ the asymptotic wave function of the form

$\varphi_\pi(x, \infty) = 6x(1-x)$. Moreover, the pion transition form factor has been measured by the CELLO [4] and, recently, the CLEO collaborations [5]. A theoretical analysis of PDA based on these data and light-cone sum rules has been undertaken [6], showing that at $Q = 2.4$ GeV PDA is neither asymptotic, nor possesses the double-hump structure [7] proposed in early works [8] [49].

The pion distribution amplitude has been evaluated with QCD sum rules [9, 10, 11, 12, 13, 14], in standard [15] (only the second ξ -moment) and transverse lattice approaches [16, 17, 18], and in chiral quark models [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. In chiral quark models the results are not always compatible to each other, and even their interpretation has not always been the same. While in some cases there are problems with chiral symmetry and proper normalization [20, 21, 25], in other cases [22, 23, 24, 25, 26, 28, 29] it is not clear how to associate the scale at which the model is defined, necessary to define the starting point for the QCD evolution. Nevertheless, there is a precise way to identify the low energy scale, Q_0 , at which the model is defined, namely the one at which the quarks carry 100% of the total momentum [30, 31]. The fact that several calculations [20, 21, 23, 24, 25, 26, 28] produce a PDA strongly resembling the asymptotic form suggests that their working scale is already large, and the subsequent QCD evolution becomes unnecessary, or numerically insignificant. This also tacitly assumes that these models already incorporate the QCD radiative corrections.

In the present paper we compute the pion distribution amplitude and the pion light-cone wave function within

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the Nambu–Jona-Lasinio (NJL) model [32, 33] in a semi-bosonized form using the Pauli-Villars (PV) regularization method [34]. This method has been introduced in Refs. [35, 36] in the context of chiral perturbation theory, as well as for chiral solitons. From the point of view of the NJL model the study of exclusive processes becomes interesting in its own right. Although factorization holds beyond doubt in QCD, it is far from obvious that any of the regularization schemes used to make a low-energy model well defined is compatible with factorization. In addition, we want to determine what is the low-energy scale, Q_0 , the model corresponds to. Here we obtain it with help of the analysis of PDA and compare it to the Q_0 obtained in deep inelastic scattering (DIS) from the corresponding parton distribution function of the pion (PDF).

To a large extent our treatment of PDA parallels the calculation of PDF carried out in previous works [37, 38, 39]. There, it has been argued that for inclusive processes, such as in deep inelastic scattering, by far the most convenient regularization scheme is the Pauli-Villars (PV) method. Such a regularization allows the extraction of the leading-twist contribution to the forward virtual Compton amplitude which possesses proper support and normalization. The relevance of regularization in chiral quark models should not be underestimated; it is not evident what is the most convenient way to cut-off high energies in such a way that most features of QCD are retained. Those include chiral symmetry, gauge invariance, and scaling properties. The main outcome of the calculation presented in Ref. [37] was that, at the scale Q_0 at which the model is defined, the valence PDF is a constant equal to one,

$$q(x, Q_0) = \bar{q}(1-x, Q_0) \equiv V_\pi(x, Q_0)/2 = 1. \quad (1)$$

After QCD evolution at leading order (LO), impressive agreement with the analysis of Ref. [40] at the reference scale $Q = 2\text{GeV}$ has been achieved. At this scale the valence quarks carry 47% of the total momentum. This implies a rather low scale Q_0 , as suggested by the evolution ratio $\alpha(2\text{GeV})/\alpha(Q_0) = 0.15$ relevant at leading order. For $\alpha(2\text{GeV}) = 0.32$ listed in the PDG [41], and for the evolution with three flavors, this corresponds to [50]

$$Q_0 = 313\text{ MeV}, \quad \alpha(Q_0) = 2.14 \quad (2)$$

(see Ref. [37] for details). The low scales are confirmed by the next-to-leading (NLO) analysis of Ref. [39], with the NLO effects small compared to the LO ones [51]. Motivated by this success, in the present paper we investigate whether the evolution ratio and the values (2) found in deep inelastic scattering are compatible with the values extracted from a similar analysis of PDA at LO in the same model (NJL) with the same (PV) regularization. This is the main objective of this work.

In the NJL model PDA has already been estimated by several authors [23, 24, 28]. The work of Refs. [23, 24] uses the Brodsky-Lepage cut-off regularization as suggested by the light-front quantization formalism. As a

consequence, the asymptotic form $\varphi(x, Q_0) = 6x(1-x)$ is obtained without any additional evolution. On the other hand, the same regularization yields PDF of the form $xV_\pi(x, Q_0) \sim 6x^2(1-x)$ [24, 42] which is far from the asymptotic value $xV_\pi(x, \infty) = x\delta(x) = 0$. This is a rather puzzling result, which may have to do with subtleties of introducing a regularization in the light-cone quantization method (see also Ref. [42]). For that reason we prefer to use a manifestly covariant formalism, where chiral symmetry can be easily implemented in presence of the regularization. In Ref. [28] PDA has been extracted from the transition form factor by examining the asymptotic behavior for large photon virtualities. This requires introducing a regularization for an abnormal parity process which also modifies the chiral anomaly, and hence, for typical parameter values [43], the $\pi^0 \rightarrow \gamma\gamma$ decay rate is reduced by 40% of the current algebra value. Our approach is free of such problems.

II. THE NAMBU–JONA-LASINIO MODEL

For the reader's convenience we briefly review the NJL model in such a way that our results can be easily stated. The $SU(2)$ NJL Lagrangian in the Minkowski space is given by [32, 33]

$$\mathcal{L}_{\text{NJL}} = \bar{q}(i\cancel{\partial} - M_0)q + \frac{G}{2}((\bar{q}q)^2 + (\bar{q}\vec{\tau}i\gamma_5 q)^2) \quad (3)$$

where $q = (u, d)$ represents a quark spinor with N_c colors, $\vec{\tau}$ are the Pauli isospin matrices, M_0 stands for the current quark mass, and G is the coupling constant. In the limiting case of the vanishing M_0 the action is invariant under the global $SU(2)_R \otimes SU(2)_L$ transformations. With help of bosonization, the vacuum-to-vacuum transition amplitude in presence of external vector and axial-vector currents, (v, a) , can be written as the path integral

$$\begin{aligned} \langle 0 | \text{Texp} \left\{ i \int d^4x [\bar{q}(\cancel{\partial} + \cancel{a}\gamma_5) q] \right\} | 0 \rangle \\ = \int D\Sigma D\vec{\Pi} \exp\{iS\}. \end{aligned}$$

The following Dirac operators

$$\begin{aligned} iD &= i\cancel{\partial} - M_0 - (\Sigma + i\gamma_5\vec{\tau} \cdot \vec{\Pi}) + \cancel{a} + \cancel{a}\gamma_5, \\ iD_5 &= -i\cancel{\partial} - M_0 - (\Sigma - i\gamma_5\vec{\tau} \cdot \vec{\Pi}) + \cancel{a} - \cancel{a}\gamma_5, \end{aligned}$$

are introduced. The fields $(\Sigma, \vec{\Pi})$ are dynamical, internal bosonic scalar-isoscalar and pseudoscalar-isovector fields, which after suitable renormalization can be interpreted as the physical σ and pion fields. The PV-regularized normal parity (γ_5 -even) contribution to the effective action is [35, 36]

$$\begin{aligned} S_{\text{even}} = -\frac{iN_c}{2} \sum_i c_i \text{tr} \log(DD_5 + \Lambda_i^2 + i\epsilon) \\ - \frac{1}{2G} \int d^4x (\Sigma^2 + \vec{\Pi}^2), \quad (4) \end{aligned}$$

with tr denoting the trace in the Dirac and isospin space. In general, we assume n PV subtractions, with the conditions $\sum_{i=0}^n c_i \Lambda_i^{2k} = 0$ for $k = 0, \dots, n$, and with $c_0 = 1$, $\Lambda_0 = 0$. At least two subtractions ($n = 2$), which is the case used throughout this paper, are needed to regularize the quadratic divergence. The abnormal parity (γ_5 -odd) contribution to the effective action is

$$S_{\text{odd}} = -\frac{iN_c}{2} \{ \text{tr} \log(D^2) - \text{tr} \log(D_5^2) \} \quad (5)$$

Notice that no explicit finite cut-off regularization is introduced in the abnormal parity contribution, as demanded by a proper reproduction of the QCD chiral anomaly. This subtle and important point has been discussed in detail in Ref. [44].

Any mesonic correlation function can be obtained from this gauge-invariantly regularized effective action by a suitable functional differentiation with respect to the relevant external fields. In practice, one usually works in the formal limit large N_c , in other words, at the one-quark-loop level. To fix the parameters in the PV-regularized NJL model we proceed as usual (see, *e.g.*, Ref. [36]). The effective potential leads to dynamical chiral symmetry breaking, thereby yielding a dynamical quark mass, M , and condensates given by

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = -\frac{M - M_0}{2G} = 4N_c M I_2, \quad (6)$$

where the quadratically-divergent integral, I_2 , is defined as

$$\begin{aligned} I_2 &= i \int \frac{d^4 k}{(2\pi)^4} \sum_i c_i \frac{1}{(-k^2 + M^2 + \Lambda_i^2 - i\epsilon)} \\ &= \frac{1}{(4\pi)^2} \sum_i c_i (\Lambda_i^2 + M^2) \log(\Lambda_i^2 + M^2). \end{aligned} \quad (7)$$

The calculation of the relevant correlation function yields for the pion mass

$$m_\pi^2 = \frac{2I_2}{F(m_\pi^2)} \frac{M_0}{M - M_0}. \quad (8)$$

The pion weak-decay constant, f_π , and the pion-quark coupling constant, $g_{\pi qq}$, are given by

$$\begin{aligned} f_\pi &= 4N_c M F(m_\pi^2) g_{\pi qq}, \\ \frac{1}{g_{\pi qq}^2} &= 4N_c \frac{d}{dp^2} \left\{ p^2 F(p^2) \right\} \Big|_{p^2=m_\pi^2}, \end{aligned} \quad (9)$$

respectively. We have introduced the following short-hand notation:

$$F(p^2) = \int_0^1 dx F(p^2, x), \quad (11)$$

where, in terms of the PV-regularized one-loop integrals,

$$\begin{aligned} F(p^2, x) &= -i \int \frac{d^4 k}{(2\pi)^4} \sum_i \times \\ &\quad \frac{1}{c_i [-k^2 - x(1-x)p^2 + M^2 + \Lambda_i^2 - i\epsilon]^2} \\ &= -\frac{1}{(4\pi)^2} \sum_i c_i \log [M^2 + \Lambda_i^2 - x(1-x)p^2]. \end{aligned} \quad (12)$$

The function F in an obvious manner satisfies the symmetry relation $F(p^2, x) = F(p^2, 1-x)$. In the case of two subtractions, and in the limit $\Lambda_1 \rightarrow \Lambda_2 \equiv \Lambda$ used in this paper, we have $\sum_i c_i f(\Lambda_i^2) = f(0) - f(\Lambda^2) + \Lambda^2 f'(\Lambda^2)$. In the numerical analysis of this paper we work in the strict chiral limit, with $M_0 = 0$. The parameters are fixed as usual; we adjust the cut-off, Λ , in order to reproduce the physical pion weak-decay constant, $f_\pi = 93.3$ MeV. The coupling constant, G , is traded for the constituent quark mass, M , which remains the only free parameter of the model. In our study of the pion light-cone wave function we use two sets, which cover the range used in other phenomenological applications of the model: $M = 280$ MeV, $\Lambda = 871$ MeV (case of Ref. [37]), and $M = 350$ MeV, $\Lambda = 770$ MeV. These give the quark condensate equal to $\langle \bar{u}u + \bar{d}d \rangle = -(290 \text{ MeV})^3$ and $-(271 \text{ MeV})^3$, respectively. As we shall see, the results are insensitive to the choice of parameters.

III. PION LIGHT-CONE WAVE FUNCTION AND PION DISTRIBUTION AMPLITUDE

The pion light-cone wave function (the axial-vector component) is defined as the low-energy matrix element [52]

$$\begin{aligned} \Psi_\pi(x, \vec{k}_\perp) &= -\frac{i\sqrt{2}}{4\pi f_\pi} \int d\xi^- d^2 \xi_\perp e^{i(2x-1)\xi^- p^+ - \xi_\perp \cdot k_\perp} \times \\ &\quad \langle \pi^+(p) | \bar{u}(\xi^-, \xi_\perp) \gamma^+ \gamma_5 d(0) | 0 \rangle. \end{aligned} \quad (13)$$

where $p^\pm = m_\pi$ and $\vec{p}_\perp = 0$. The pion distribution amplitude is defined as

$$\varphi_\pi(x) = \int d^2 k_\perp \Psi_\pi(x, \vec{k}_\perp) \quad (14)$$

Formally, in the momentum space, Eq. (13) corresponds to integration over the quark momenta in the loop integral used in the evaluation of f_π , but with $k^+ = p^+ x = m_\pi x$ and k_\perp fixed. Thus, with the PV method and after working out the Dirac traces, we have to compute

$$\begin{aligned} \Psi_\pi(x, \vec{k}_\perp) &= -\frac{2iN_c M g_{\pi qq}}{f_\pi} \int \frac{dk^+ dk^-}{(2\pi)^4} \times \\ &\quad \frac{\delta(k^+ - xp^+)}{m_\pi x(1-x)} \sum_j c_j \times \\ &\quad \frac{1}{k^- - m_\pi + \frac{\vec{k}_\perp^2 + M^2 + \Lambda_j^2 + i0^+}{m_\pi(1-x)}} \frac{1}{k^- - \frac{\vec{k}_\perp^2 + M^2 + \Lambda_j^2 + i0^+}{m_\pi x}}, \end{aligned} \quad (15)$$

where the location of the poles in the k^- variable has been explicitly displayed. Evaluating the k^- integral gives the pion LC wave function in the NJL model with the PV regularization:

$$\Psi_\pi(x, k_\perp) = \frac{4N_c M g_{\pi qq}}{16\pi^3 f_\pi} \sum_j c_j \times \frac{1}{k_\perp^2 + \Lambda_j^2 + M^2 - x(1-x)m_\pi^2}. \quad (16)$$

The function is properly normalized,

$$\int d^2 k_\perp dx \Psi_\pi(x, k_\perp) = 1, \quad (17)$$

and satisfies the crossing relation

$$\Psi_\pi(x, \vec{k}_\perp) = \Psi_\pi(1-x, \vec{k}_\perp). \quad (18)$$

For $m_\pi \neq 0$ it is non-factorizable in the k_\perp and x variables. Integrating with respect to k_\perp yields the pion distribution amplitude,

$$\varphi_\pi(x) = 4N_c M F(m_\pi^2, x) \frac{g_{\pi qq}}{f_\pi}. \quad (19)$$

The crossing property, $\varphi_\pi(x) = \varphi_\pi(1-x)$ follows trivially, and Eq. (9) gives the correct normalization, namely $\int dx \varphi_\pi(x) = 1$.

As a consequence of the PV condition with two subtractions one has, for large k_\perp ,

$$\Psi_\pi(x, k_\perp) \rightarrow \frac{4N_c M^2}{16\pi^3 f_\pi^2} \frac{\sum_i c_i \Lambda_i^4}{k_\perp^6}, \quad (20)$$

which gives a finite normalization and a finite second transverse moment,

$$\langle k_\perp^2 \rangle = \int d^2 k_\perp \int_0^1 dx \Psi_\pi(x, k_\perp) k_\perp^2 \quad (21)$$

In the chiral limit, $m_\pi = 0$, one can use the Goldberger-Treiman relation for the constituent quarks, $g_{\pi qq} f_\pi = M$. Then $f_\pi^2 = 4N_c M^2 F(0)$, which gives the very simple formulas

$$\Psi_\pi(x, k_\perp) = \frac{4N_c M^2}{16\pi^3 f_\pi^2} \sum_i c_i \frac{1}{k_\perp^2 + \Lambda_i^2 + M^2}, \quad (22)$$

$$\varphi_\pi(x) = 1, \quad (23)$$

$$\langle \vec{k}_\perp^2 \rangle = -\frac{M \langle \bar{u}u \rangle}{f_\pi^2}. \quad (24)$$

In the chiral limit $\Psi_\pi(x, \vec{k}_\perp)$ becomes trivially factorizable, since it is independent of x . A remarkable feature is that the last two relations, Eq. (23) and Eq. (24), are independent of the PV regulators. A similar situation has also been encountered when computing PDF in the chiral limit [37]; it was a constant equal to one, regardless on the details of the PV regulator. We will show below that

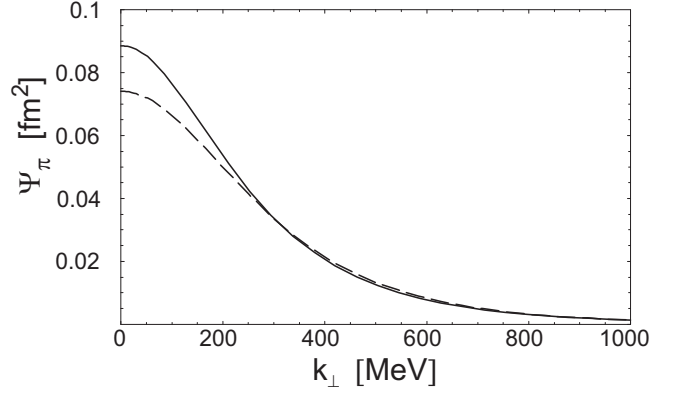


FIG. 1: The pion light-cone wave function in the chiral limit, evaluated in the Nambu–Jona-Lasinio with the Pauli-Villars regularization with two subtractions and with the constituent quark mass $M = 280\text{MeV}$ (solid line) and 350 MeV (dashed line), plotted as a function of the transverse momentum k_\perp . The wave function does not depend on x . The normalization is such that $\int d^2 \vec{k}_\perp \Psi_\pi(x, k_\perp) = \varphi_\pi(x) = 1$. The second transverse moment is $\langle \vec{k}_\perp^2 \rangle = -M \langle \bar{u}u \rangle / f_\pi^2 = (625\text{MeV})^2$ for $M = 280\text{MeV}$ and $(634\text{MeV})^2$ for $M = 350\text{MeV}$. The scale relevant for the calculation, as inferred from the QCD evolution [37], is $Q_0 = 313\text{ MeV}$.

by putting together Eq. (23) and the results of Ref. [37] an interesting relation follows.

Higher transverse moments diverge if one restricts the number of Pauli-Villars subtractions to two, but Eq. (23) and Eq. (24) remain still valid if more subtractions are considered.

In Fig. (1) we show the k_\perp -dependence of the light-cone pion wave function in the chiral limit (finite pion mass corrections turn out to be tiny, at the level of a few %) for the PV regularization with two subtractions, and with $M = 380\text{MeV}$ and 350 MeV . For these values we get the transverse moment $\langle \vec{k}_\perp^2 \rangle = (625\text{MeV})^2$, and $(634\text{MeV})^2$, respectively. This value is about a factor of two larger than the one found in Ref. [25], namely $(430\text{MeV})^2$, and a factor of four higher than the findings of Ref. [45], $(316\text{MeV})^2$, at the scale at which $\alpha/\pi \sim 0.1$, i.e. $Q \sim 1-2\text{GeV}$. As we shall see below, a part of the discrepancy can be attributed to the QCD radiative corrections.

In non-local versions of the chiral quark model, where a momentum-dependent mass function is introduced as a physically motivated regulator, the trend to produce a constant PDA has also been observed if the constant mass limit is considered [20, 21, 25]. In those models such a limit effectively corresponds to removing the regulator, against the original spirit of the model. Unfortunately, for the genuine non-local case those calculations violate proper normalization of PDA, because the employed currents do not comply with the necessary Ward identities required by chiral symmetry. The problem has been addressed in Ref. [26], where it has been found that about a third of the normalized PDA comes from

the non-local currents. For a Gaussian mass function there is a clear flattening of $\varphi_\pi(x)$ in the central region of $0.2 \leq x \leq 0.8$ [29].

We stress that our result, Eq. (23), holds true without removing the Pauli-Villars regulator and is in harmony with chiral symmetry, since the starting point was the normal parity action, which by construction preserves chiral symmetry. Obviously, the fact that our final answer does not depend on the form of the PV regulators used makes any subsequent manipulation with the regulators fully irrelevant.

Another point is that PDA from Eq. (23) and PDF from Eq. (1) yield the relation $\varphi_\pi(x) = V_\pi(x)/2$ valid at a low scale Q_0 . It is noteworthy that in the framework of QCD sum rules the same identity between PDA and PDF has also been obtained [11] at some scale, although there the asymptotic form for PDA was assumed without the QCD evolution, while PDF was obtained by QCD evolution. We will show below that if evolution is undertaken for both PDA and PDF at the same low energy scale, an overall consistent picture arises.

IV. QCD EVOLUTION

The comparison of the leading-twist PDA to high-energy experimental data requires, like for PDF, the inclusion of radiative logarithmic corrections through the QCD evolution [2, 3]. For the pion distribution amplitude this is done in terms of the Gegenbauer polynomials, by interpreting our low-energy model result as the initial condition. For clarity we work in the chiral limit, hence

$$\varphi_\pi(x, Q_0) = 1. \quad (25)$$

Then, the LO-evolved distribution amplitude reads [2, 3]

$$\varphi_\pi(x, Q) = 6x(1-x) \sum_{n=0}^{\infty'} C_n^{3/2}(2x-1) a_n(Q), \quad (26)$$

where the prime indicates summation over even values of n only. The matrix elements, $a_n(Q)$, are the Gegenbauer moments given by

$$a_n(Q) = \frac{2}{3} \frac{2n+3}{(n+1)(n+2)} \left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{\gamma_n^{(0)}/(2\beta_0)} \times \int_0^1 dx C_n^{3/2}(2x-1) \varphi_\pi(x, Q_0), \quad (27)$$

with $C_n^{3/2}$ denoting the Gegenbauer polynomials, and

$$\begin{aligned} \gamma_n^{(0)} &= -2C_F \left[3 + \frac{2}{(n+1)(n+2)} - 4 \sum_{k=1}^{n+1} \frac{1}{k} \right], \\ \beta_0 &= \frac{11}{3} C_A - \frac{2}{3} N_F, \end{aligned} \quad (28)$$

with $C_A = 3$, $C_F = 4/3$, and N_F being the number of active flavors, which we take equal to three [53]. With

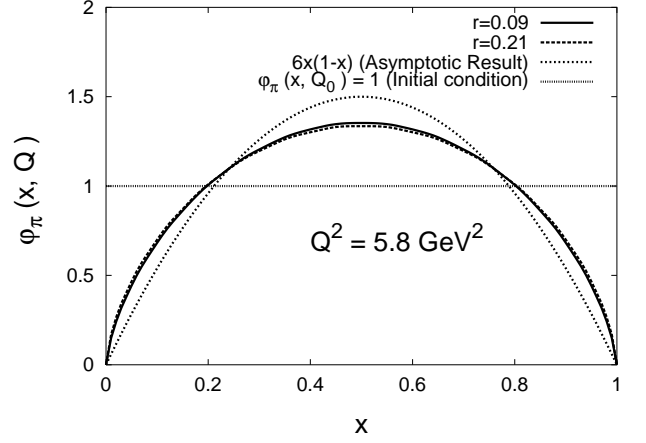


FIG. 2: The pion distribution amplitude in the chiral limit evolved to the scale $Q^2 = (2.4\text{GeV})^2$. The two values for the evolution ratio $r = \alpha(Q)/\alpha(Q_0)$ reflect the uncertainties in the values of Ref. [6] based on an analysis of the CLEO data. We also show the unevolved PDA, $\varphi_\pi(x, Q_0) = 1$, and the asymptotic PDA, $\varphi_\pi(x, \infty) = 6x(1-x)$.

our constant amplitude (25) we get immediately

$$\int_0^1 dx C_n^{3/2}(2x-1) \varphi_\pi(x, Q_0) = 1. \quad (29)$$

Thus, for a given value of Q we may predict PDA. We need, however, to know what the initial scale Q_0 is, or, equivalently, to know the evolution ratio $r = \alpha(Q)/\alpha(Q_0)$. The fitting procedure of Ref. [6] yields $a_2(2.4\text{GeV}) = 0.12 \pm 0.03$ (with the assumption $a_k = 0$, $k > 2$). We treat this as experimental input, and then with help of Eqs. (27,29) we get for the evolution ratio

$$\alpha(Q = 2.4\text{GeV})/\alpha(Q_0) = 0.15 \pm 0.06. \quad (30)$$

which at LO implies $Q_0 = 322 \pm 45\text{MeV}$, a value compatible within errors with (2).

The fit of Ref. [6] with non-zero a_4 yields $a_2 = 0.19 \pm 0.04 \pm 0.09$ and $a_4 = -0.14 \pm 0.03 \mp 0.09$. The central value of a_2 would imply, according to our prescription, the evolution ratio of 0.31, and, correspondingly, $Q_0 = 0.47^{+0.51}_{-0.19} \text{ GeV}$, a much larger central value than (2), but with very large errors. For that reason, in the numerical studies below we use the value (30) for the evolution ratio.

We can now predict the following lowest-order coefficients:

$$\begin{aligned} a_4(2.4\text{GeV}) &= 0.044 \pm 0.016 \\ a_6(2.4\text{GeV}) &= 0.023 \pm 0.010 \\ a_8(2.4\text{GeV}) &= 0.014 \pm 0.006 \\ a_{10}(2.4\text{GeV}) &= 0.009 \pm 0.005 \end{aligned} \quad (31)$$

For the sum of the Gegenbauer coefficients we get the estimate

$$\begin{aligned} \sum_{n=2}^{\infty} a_n(Q = 2.4\text{GeV}) &= \int_0^1 dx \frac{\varphi_\pi(x, Q = 2.4\text{GeV})}{6x(1-x)} - 1 \\ &= 0.25 \pm 0.10 \end{aligned} \quad (32)$$

where the uncertainties correspond to the uncertainties in Eq. (30).

The leading-twist contribution to the pion transition form factor is, at the LO in the QCD evolution [1], equal to

$$\frac{Q^2 F_{\gamma^* \rightarrow \pi \gamma}(Q)}{2f_\pi} \Big|_{\text{twist}-2} = \int_0^1 dx \frac{\varphi_\pi(x, Q)}{6x(1-x)} \quad (33)$$

The experimental value obtained in CLEO [5] for the full form factor is $Q^2 F_{\gamma^*, \pi \gamma}(Q)/(2f_\pi) = 0.83 \pm 0.12$ at $Q^2 = (2.4\text{GeV})^2$. Our value for the integral, 1.25 ± 0.10 , overestimates the experimental result, although at the 2σ -confidence level both numbers are compatible. Taking into account the fact that we have not included neither NLO effects nor an estimate of higher-twist contributions, the result is quite encouraging.

In Fig. 2 we show our PDA evolved to $Q = 2.4\text{GeV}$, for two values of the evolution ratio, which reflect the uncertainties from Eq. (30). We also show the initial and the asymptotic PDA's. It is interesting to note that after evolution our results closely resemble those found in transverse lattice approaches [16, 17, 18]. In particular, we get for the second ξ -moment ($\xi = 2x - 1$),

$$\begin{aligned} \langle \xi^2 \rangle &= \int_0^1 dx \varphi_\pi(x, Q = 2.4\text{GeV}) (2x - 1)^2 \\ &= 0.040 \pm 0.005, \end{aligned} \quad (34)$$

to be compared with $\langle \xi^2 \rangle = 0.06 \pm 0.02$ obtained in the standard lattice QCD for $Q = 1/a = 2.6 \pm 0.1\text{GeV}$ [15]. From the PDF calculation at LO of Ref. [37] we estimate that if the momentum fraction carried by the valence quarks at $Q = 2\text{GeV}$ is $0.47 \pm 0.02\%$, then Q_0 is such that $\alpha(Q_0) = 2.14$, and the evolution ratio at $Q = 2\text{GeV}$ is $r = 0.15$. Then, for $Q = 2.4\text{GeV}$ we get $r = 0.14$ from the analysis of PDF, a value compatible, within uncertainties, with the present calculation, Eq. (30). This is a crucial finding, showing the consistency of the results obtained in our approach.

One might worry that the starting condition (25) does not satisfy the end-point vanishing behavior and therefore cannot be expanded in terms of the Gegenbauer polynomials. This is true, provided one insists on uniform pointwise convergence. However, the Gegenbauer polynomials form a complete set in the space of square-summable functions, hence convergence may be understood in a weak sense [54]. The slow convergence is reflected by the fact that in Fig. 2 at least 30-100 Gegenbauer polynomials are needed for evolution ratios $r = 0.9 - 0.21$ respectively. The convergence at the mid-point, $x = 1/2$, is improved, since the series for $\varphi(x, Q)$

is sign-alternating. At the end-points, $x = 0, 1$, the series diverges, since $C_{2k}^{3/2}(\pm 1) = \frac{1}{2}(2k+1)(2k+2)$, which means that the convergence in Eq. (26) is not uniform. In order to analyze the behavior close to the end-points in a greater detail we consider the large- n contribution to Eq. (26). We have

$$\left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{\gamma_n^{(0)}/(2\beta_0)} \rightarrow n^{-\frac{4C_F}{\beta_0} \ln \frac{\alpha(Q)}{\alpha(Q_0)}}, \quad (35)$$

hence, for $Q \rightarrow Q_0$, $Q > Q_0$, and with $x \rightarrow 0$ (recall that the function is symmetric under $x \rightarrow 1 - x$), we obtain

$$\varphi_\pi(x \rightarrow 0, Q) \rightarrow 8x\zeta \left(\frac{4C_F}{2\beta_0} \ln \frac{\alpha(Q)}{\alpha(Q_0)} + 1 \right), \quad (36)$$

where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is the Riemann ζ function, and $\zeta(1) = \sum_{n=1}^{\infty} n^{-1} = \infty$. Thus the slope of the evolved PDA at the end-points becomes steeper and steeper as $Q \rightarrow Q_0$.

The QCD evolution also influences the value of the transverse moment. According to the work of Ref. [45], $\langle \vec{k}_\perp^2 \rangle$ can be expressed as $\langle \vec{k}_\perp^2 \rangle = 5m_0^2/36$, where $m_0^2 = \langle \bar{q}\sigma^{\mu\nu}F_{\mu\nu}q \rangle / \langle \bar{q}q \rangle$ is the ratio between the quark-gluon and quark condensates. The quantity m_0^2 is scale dependent and has been estimated to be $m_0^2(1\text{GeV}) = 0.8 \pm 0.2\text{GeV}^2$ [46]. Using the corresponding anomalous dimensions, 4 for $\langle \bar{q}q \rangle$ and $-2/3$ for $\langle \bar{q}\sigma^{\mu\nu}F_{\mu\nu}q \rangle$ [47], yields

$$\begin{aligned} \frac{\langle k_\perp^2 \rangle_Q}{\langle k_\perp^2 \rangle_{Q_0}} &= \left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{(4+2/3)/\beta_0} \\ &= \left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{14/(33-2N_f)}. \end{aligned} \quad (37)$$

For $N_F = 3$ this scale dependence can be seen in Fig. (3). For the values $Q = 1 - 2\text{GeV}$ one gets a reduction factor of $0.37 - 0.45$ for the ratio (37), and $\langle k_\perp^2 \rangle_Q = (430\text{MeV})^2 - (380\text{MeV})^2$ for the second transverse moment, somewhat higher than the QCD sum rules estimate based on Ref. [45], $(316\text{MeV})^2$, or on Ref. [46], $(333 \pm 40\text{MeV})^2$.

V. THE RELATION TO DEEP INELASTIC SCATTERING

As we have already stated in Eq. (1), the valence PDF for the pion in the chiral limit has also been found to be a constant equal to one [37]. At LO the non-singlet evolution of the PDF moments is quite similar to that of

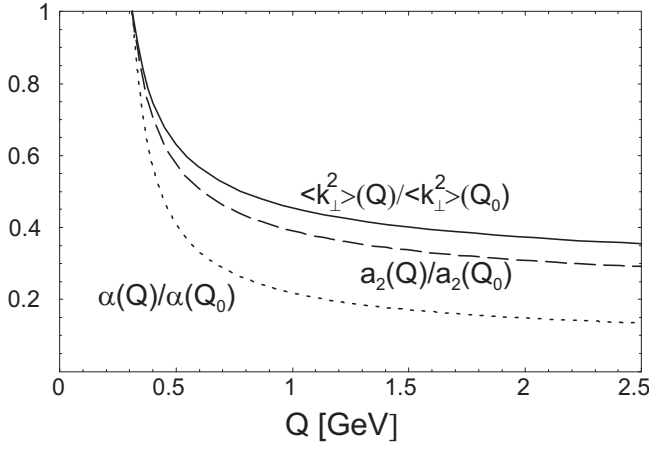


FIG. 3: Dependence of the second transverse moment of the pion light-cone wave function, $\langle k_{\perp}^2 \rangle_Q / \langle k_{\perp}^2 \rangle_{Q_0}$ (solid line), the second Gegenbauer moment $a_2(Q)/a_2(Q_0)$ of the pion distribution amplitude (dashed line), and the evolution ratio $r = \alpha(Q)/\alpha(Q_0)$ (dotted line), plotted as functions of the scale Q . The leading-order QCD evolution is applied. All quantities are relative to their values at the low energy scale, $Q_0 = 313$ MeV, at which the momentum fraction carried by the quarks equals unity [37], according to the prescription that in a quark model Q_0 is defined by the condition $\langle x V_{\pi}(x, Q_0) \rangle = 1$. In our model $\alpha(Q_0) = 2.14$, $a_2(Q_0) = 7/18$, and $\langle k_{\perp}^2 \rangle_{Q_0} = (625 \text{ MeV})^2$ for $M = 280 \text{ MeV}$ and $(634 \text{ MeV})^2$ for $M = 350 \text{ MeV}$ and in the chiral limit.

the Gegenbauer moments of PDA, Eq. (26), namely

$$\int_0^1 dx x^n V_{\pi}(x, Q) = \left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{\gamma_n^{(0)}/(2\beta_0)} \int_0^1 dx x^n V_{\pi}(x, Q_0) = \frac{2}{n+1} \left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{\gamma_n^{(0)}/(2\beta_0)}. \quad (38)$$

Thus, for $n = 2$, one obtains

$$\frac{a_2(Q)}{a_2(Q_0)} = \frac{\langle x^2 V_{\pi}(x, Q) \rangle}{\langle x^2 V_{\pi}(x, Q_0) \rangle} = \left(\frac{\alpha(Q)}{\alpha(Q_0)} \right)^{\gamma_2^{(0)}/(2\beta_0)}. \quad (39)$$

For $N_F = 3$ this scale dependence for the ratios can be looked up in Fig. (3). Using $\langle x^2 V_{\pi}(x, Q_0) \rangle = 2/3$ and $a_2(Q_0) = 7/18$ yields

$$\frac{a_2(Q)}{\langle x^2 V_{\pi}(x, Q) \rangle} = \frac{7}{12}, \quad (40)$$

hence $a_2(2 \text{ GeV}) = 0.12 \pm 0.01$ for $\langle x^2 V_{\pi} \rangle = 0.20 \pm 0.01$ [40] and $a_2(2 \text{ GeV}) = 0.10 \pm 0.01$ for $\langle x^2 V_{\pi} \rangle = 0.17 \pm 0.01$ [48].

One can combine Eqs. (1,23,26,39) to obtain the following very interesting LO relation that holds in the considered model:

$$\frac{\varphi_{\pi}(x, Q)}{6x(1-x)} - 1 = \int_0^1 dy K(x, y) V_{\pi}(y, Q), \quad (41)$$

where the kernel K is independent of Q^2 , and is given by

$$K(x, y) = \sum_{n=2}^{\infty} \frac{(2n+3)}{3(n+2)} C_n^{3/2} (2x-1) y^n. \quad (42)$$

In general, the relation (41) holds in any model where PDA and PDF are simultaneously equal to unity at some scale Q_0 , and are subsequently evolved at LO. Physically, Eq. (41) simply tells us that the departure of PDA at a given Q^2 from the asymptotic form is proportional to a weighted integral of PDF at the same Q . Clearly, $\varphi_{\pi}(x, Q) \rightarrow 6x(1-x)$ if $V_{\pi}(x, Q) \rightarrow 2\delta(x)$ or, equivalently, $xV_{\pi}(x, Q) \rightarrow 0$, since $K(x, 0) = 0$. Roughly speaking, in the present model the pion distribution function is as close to the asymptotic value as the non-singlet parton distribution. A remarkable feature of relation (41) is that it binds matrix elements related to exclusive (PDA) and to inclusive (PDF) processes.

In order to evaluate the kernel we use the symmetrized generating function of the Gegenbauer polynomials,

$$G(x, y) = \sum_{n=2}^{\infty} C_n^{3/2} (2x-1) y^n = \frac{1}{2} \left\{ R_+^{-3/2} + R_-^{-3/2} \right\} - 1$$

$$R_{\pm} = 1 \mp 2(2x-1)y + y^2, \quad (43)$$

whence one can obtain

$$K(x, y) = \frac{2}{3} G(x, y) - \frac{1}{3y^2} \int_0^y dy' y' G(x, y'). \quad (44)$$

The integrals can be worked out to yield the final result

$$K(x, y) = \frac{1}{24R_+^{3/2} y^2 (x-1)x} \times$$

$$[8(x-1)xy^2 + R_+((2x-1)y-1) + 2\sqrt{R_+}(x-1)xy^2(1+(2-4x)y+y^2) + R_+^{\frac{3}{2}}(1-8(x-1)xy^2)] - (y \leftrightarrow -y) \quad (45)$$

To test the success of Eq. (41) we need some input for $V_{\pi}(x, Q)$. However, taking into account the fact that the agreement of the evolved valence PDF, $V_{\pi}(x, Q)$ with the parameterization of Ref. [40] at $Q^2 = 4 \text{ GeV}^2$ is almost perfect [37, 39], and that the results are almost insensitive to the evolution ratio, $\alpha(Q)/\alpha(Q_0)$, Fig. 2 can be regarded as a direct prediction of Eq. (41) taking Ref. [40] as input for $V_{\pi}(x, Q)$. A further consequence of Eq. (41) may be obtained by integrating with respect to x and performing the sum over n . Through the use of Eq. (32) we get

$$\sum_{n=2}^{\infty} a_n(Q) = \int_0^1 dy \kappa(y) V_{\pi}(y, Q) \quad (46)$$

where

$$\kappa(y) = \int_0^1 dx K(x, y) = \sum_{n=2}^{\infty} \frac{(2n+3)}{3(n+2)} y^n$$

$$= \frac{3y^2+1}{6(1-y^2)} + \frac{\log(1-y) + \log(1+y)}{6y^2}. \quad (47)$$

Notice that, for $Q \rightarrow \infty$ we get $V_\pi(x, Q) \rightarrow 2\delta(x)$ and since $\kappa(y) = 7y^2/12 + \mathcal{O}(y^4)$ one gets $\sum_{n=2}^{\infty} a_n(Q) \rightarrow 0$, as expected. Finally, using the parameterization of Ref. [40] we get [55]

$$\sum_{n=2}^{\infty} a_n(2\text{GeV}) = 0.25 \pm 0.03, \quad (48)$$

a value perfectly compatible with Eq. (32) although with smaller uncertainties [56]. Again, this verifies the consistency of our approach.

VI. CONCLUSIONS

We summarize our points. We have computed the light-cone pion wave function and the pion distribution amplitude in the Nambu–Jona-Lasinio model. To this end, and to comply with previous results regarding the parton distribution functions, we have used the Pauli-Villars regularization method in such a way that chiral symmetry, gauge invariance, and relativistic invariance are preserved. As a result, we find that in the chiral limit the pion distribution amplitude, computed as a low energy matrix element of an appropriate operator, is a constant equal to one, $\varphi_\pi(x) = 1$, and the second transverse moment of the pion light-cone wave function is $\langle \vec{k}_\perp^2 \rangle = -M \langle \bar{u}u \rangle / f_\pi^2$, with M denoting the constituent quark mass. Both results are independent of the particular form of the Pauli-Villars regulators used. After the QCD evolution of the pion distribution am-

plitude to the experimentally accessible region we find a result still rather far away from the asymptotic form, $\varphi_\pi(x) = 6x(1-x)$, but in a good agreement with the analysis of the experimental data from the CLEO collaboration. We can determine the working momentum scale for the model to be $Q_0 = 313$ MeV, a rather low value. Moreover, the scale Q_0 obtained in this work is compatible, within experimental uncertainties, to the value obtained from the previous analysis of the parton distribution functions, carried out within exactly the same model. At the scale Q_0 the quarks carry all the momentum of the pion. Our value obtained for the second transverse moment of the pion light-cone wave function, $\langle \vec{k}_\perp^2 \rangle$, becomes, after the QCD evolution, not far from the estimates based on the QCD sum rules. Finally, we have also derived a model relation which binds the departure of the pion distribution amplitude from its asymptotic value to an integral involving the pion quark distribution function. The relation, specific to the feature of our model that at the scale Q_0 both the PDA and PDF are constant and equal to unity, has been successfully checked against the available data.

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- [49] A recent direct measurement of PDA via the di-jet production by the E791 collaboration [8] shows that at scales $Q \simeq 5 - 6$ GeV, the possible admixture of the Chernyak-Zhitnitsky wave function is rather small
- [50] In this paper we use the LO QCD evolution, where $\alpha(Q) = (4\pi/\beta_0)/\log(Q^2/\Lambda_{\text{QCD}}^2)$ with $\Lambda_{\text{QCD}} = 226$ MeV for $N_F = 3$.
- [51] Admittedly, one may worry that such a low scale as in Eq. (2) may invalidate the use of perturbation theory. We hope that the corresponding value of $\alpha/(2\pi) \sim 0.3$, which typically is the expansion parameter, is low enough for the approach to make sense.
- [52] The light-cone coordinates are defined as $k^\pm = k^0 \pm k^3$ and $d^4k = \frac{1}{2}dk^+dk^-d^2\vec{k}_\perp$
- [53] The one-loop anomalous dimension $\gamma_n^{(0)} > 0$, and $\gamma_n^{(0)} \rightarrow 8C_F \log n$ for large n , which coincides with the case of the non-singlet parton distribution functions used in Refs. [37, 39].
- [54] The condition for $\varphi_\pi(x)$ to belong to such a space is $\int_0^1 dx \frac{\varphi_\pi(x)}{x(1-x)} < \infty$. The function $\varphi_\pi(x) = 1$ does not belong to this space, but it belongs to its closure. This resembles the well-known fact that plane waves do not belong to the space of square-summable functions in the interval $-\infty < x < \infty$, but nevertheless may be approximated by square summable-functions.
- [55] This is $xV_\pi(x, Q) = A_V x^\alpha(1-x)^\beta$ with A_V such that $\langle V_\pi \rangle = 2$ and $\alpha = 0.64 \pm 0.03$ and $\beta = 1.08 \pm 0.02$ (the NA10 set) and $\beta = 1.15 \pm 0.02$ (the E615 set). Our estimate of error includes both sets.
- [56] Of course, this estimate does not include systematic uncertainties in NLO both for PDA and PDF.